# Viscous Relativistic Hydrodynamics*

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**Abstract**
I review recent and not so recent progress on formulating and numerically implementing a consistent set of relativistic equations which describe the space-time evolution of viscous relativistic fluids without violating causality.

## 1 Introduction

Ideal fluid dynamics has been used successfully to predict the collective flow patterns in Au+Au collisions at the Relativistic Heavy-Ion Collider RHIC (for a review see [1]). The ideal fluid description works well in almost central Au+Au collisions near midrapidity at top RHIC energy, but gradually breaks down in more peripheral collisions, at forward rapidity, or at lower collision energies [2]. To describe such deviations from ideal fluid dynamics quantitatively and use the experimental data to extract values or phenomenological limits for the transport coefficients of the hot and dense matter created during the collision requires the numerical implementation of dissipative relativistic fluid dynamics. Although a formulation of such a theory which avoids the longstanding problems of acausal signal propagation and other instabilities associated with the original relativistic fluid equations given by Eckart and Landau & Lifshitz, has been known for almost 30 years [3], significant progress towards its numerical implementation has only been made very recently [4, 5, 6, 7]. At this point, we are only at the very beginning of a program that will eventually apply a viscous relativistic fluid dynamical approach to heavy-ion collision data. Existing numerical implementations are (1+1)-dimensional and can only describe cylindrically symmetric transverse expansion with boost-invariant longitudinal dynamics [6, 7]. The codes are still in the process of being tested. In this contribution I try to give a pedagogical summary [8] of the theoretical background and summarize a few first results.

## 2 Ideal fluid dynamics

Before explaining the structure of the equations for causal dissipative relativistic fluid dynamics, let me quickly review the case of ideal fluid dynamics. Any fluid dynamical approach starts from the conservation laws for the conserved charges and for energy-momentum,

$$\partial_\mu N_i^\mu = 0, \quad i = 1, \ldots, k, \tag{1}$$

$$\partial_\mu T^{\mu\nu} = 0. \tag{2}$$

For simplicity I will restrict myself to $k = 1$ (say, $N_\mu =$ net baryon number current) and drop the index $i$ in (1). It must also ensure the second law of thermodynamics

$$\partial_\mu S^\mu \geq 0, \tag{3}$$

where $S^\mu$ is the entropy current. Ideal fluid dynamics follows from these equations under the assumption of local thermal equilibrium, i.e. if the microscopic collision time scale is very much shorter than any macroscopic evolution time scale such that the the underlying phase-space distribution $f(x,p)$ relaxes essentially instantaneously to a local equilibrium form (upper signs for fermions, lower signs for bosons)

$$f_{\rm eq}(x,p) = \frac{1}{e^{[p\cdot u(x)+\mu(x)]/T(x)} \pm 1}, \tag{4}$$

*Work supported by the U.S. Department of Energy under contract DE-FG02-01ER41190.

where $u^\mu(x)$ is the local fluid velocity at point $x$, $\mu(x)$ is the local chemical potential associated with the conserved charge $N$ (it enters with opposite sign in the distribution $\bar{f}$ for antiparticles), and $T(x)$ is the local temperature. Plugging this into the kinetic theory definitions

$$N^\mu(x) = \int \frac{d^3p}{E} p^\mu [f(x,p) - \bar{f}(x,p)], \tag{5}$$

$$T^{\mu\nu}(x) = \int \frac{d^3p}{E} p^\mu p^\nu [f(x,p) + \bar{f}(x,p)], \tag{6}$$

$$S^\mu(x) = -\int \frac{d^3p}{E} p^\mu [f(x,p)\ln f(x,p) \pm (1\mp f(x,p))\ln(1\mp f(x,p)) + (f \leftrightarrow \bar{f})] \tag{7}$$

leads to the ideal fluid decompositions

$$N^\mu_{\rm eq} = n\, u^\mu, \tag{8}$$

$$T^{\mu\nu}_{\rm eq} = e\, u^\mu u^\nu - p\, \Delta^{\mu\nu} \quad (\text{with } \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu), \tag{9}$$

$$S^\mu_{\rm eq} = s\, u^\mu, \tag{10}$$

where the local net charge density $n$, energy density $e$, pressure $p$ and entropy density $s$ are given by the standard integrals over the thermal equilibrium distribution function in the local fluid rest frame and are related by the fundamental thermodynamic relation

$$T\, s = p - \mu\, n + e. \tag{11}$$

Inserting Eqs. (5-7) into Eqs. (1,2) yields the relativistic ideal fluid equations shown in Eqs. (12-14) below. Using Eq. (11) together with the Gibbs-Duhem relation $dp = s\, dT + n\, d\mu$, it is easy to prove that, in the absence of shock discontinuities, these equations also conserve entropy, i.e. $\partial_\mu S^\mu = 0$.

Note that the validity of the decompositions (5-7) only requires local momentum isotropy (i.e. that in the local fluid rest frame the phase-space distribution reduces to a function of energy $E$ only, $f(x,p) = f(p{\cdot}u(x); T(x), \mu(x))$), but not that the distribution function has the specific exponential form (4) which maximizes entropy. This may have relevance in situations where the time scale for local momentum isotropization is much shorter than for thermalization (i.e. it is much easier to change the direction of the particles' momenta than their energies), with the macroscopic hydrodynamic time scale in between. In this case the local microscopic states would not maximize entropy, and the relation (11) would not hold between the quantities $e$, $p$, $n$, and $s$ defined through eqs. (5-10). Still, they would follow ideal fluid dynamical evolution since entropy production by microscopic kinetic energy-shifting processes would happen only on time scales which are large compared to the macroscopic evolution time scales.

The ideal fluid equations read (with $\theta \equiv \partial{\cdot}u$ denoting the local expansion rate)

$$\dot{n} = -n\,\theta, \tag{12}$$

$$\dot{e} = -(e+p)\,\theta, \tag{13}$$

$$\dot{u}^\mu = \frac{\nabla^\mu p}{e+p}, \tag{14}$$

where we decomposed the partial derivative $\partial^\mu = u^\mu D + \nabla^\mu$ into "longitudinal" and "transverse" components $D = u^\nu\partial_\nu$ and $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$ which in the local fluid rest frame reduce to the time derivative $\dot{f} \equiv Df$ and spatial gradient. The first two equations describe the dilution of the local baryon and energy density due to the local expansion rate $\theta$, while the third describes the acceleration of the fluid by the spatial (in the local frame) pressure gradients, with the enthalpy $e{+}p$ acting as inertia. The 5 equations (12-14) for the 6 unknown functions $n$, $e$, $p$, $u^\mu$ (remember that $u^\mu u_\mu = 1$) must be closed by supplying an *Equation of State (EOS)* $p = p(n,e)$.

## 3 Non-ideal fluid decomposition

As the hydrodynamic evolution changes the local energy and baryon density, microscopic process attempt to readjust the local phase-space distribution to corresponding new local temperatures and chemical potentials. If this does not happen sufficiently fast, the phase-space distribution will start to deviate from its local equilibrium form (4): $f(x,p) = f_{\rm eq}(p\cdot u(x); T(x), \mu(x)) + \delta f(x,p)$. The optimal values for the (readjusted) local temperature and chemical potential in the first term are fixed by imposing the "Landau matching conditions"

$$u_\mu\, \delta T^{\mu\nu} u_\nu = \int \frac{d^3p}{E}\, (u\cdot p)^2\, \delta f(x,p) = u_\mu\, \delta N^\mu = \int \frac{d^3p}{E} u\cdot p\, \delta f(x,p) = 0. \tag{15}$$

The remaining deviations $\delta f$ from local equilibrium generate additional terms in the decompositions of $N^\mu$, $T^{\mu\nu}$, and $S^\mu$:

$$\begin{aligned} N^\mu &= N^\mu_{\rm eq} + \delta N^\mu = n\, u^\mu + V^\mu, & (16)\\ T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} + \delta T^{\mu\nu} = e\, u^\mu u^\nu - (p+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu, & (17)\\ S^\mu &= S^\mu_{\rm eq} + \delta S^\mu = n\, u^\mu + \Phi^\mu. & (18) \end{aligned}$$

The new terms describe a baryon number flow $V^\mu = \Delta^{\mu\nu} N_\nu$ in the local rest frame, an energy flow $W^\mu = \frac{e+p}{n} V^\mu + q^\mu$ (where $q^\mu$ is the "heat flow vector") in the local rest frame, the viscous bulk pressure $\Pi = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} - p$ (which contributes to the trace of the energy momentum tensor), the traceless viscous shear pressure tensor $\pi^{\mu\nu} = T^{\langle\mu\nu\rangle} \equiv \left[\frac{1}{2}\left(\Delta^{\mu\sigma}\Delta^{\nu\tau} + \Delta^{\nu\sigma}\Delta^{\mu\tau}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\sigma\tau}\right] T_{\tau\sigma}$ (where the expression $\langle\mu\nu\rangle$ is a shorthand for "traceless and transverse to $u_\mu$ and $u_\nu$", as defined by the projector in square brackets), and an entropy flow vector $\Phi^\mu$ in the local rest frame.

Inserting the decompositions (15,17) into the conservation laws (1,2) yields the *non-ideal fluid equations*

$$\begin{aligned} \dot n &= -n\,\theta - \nabla\cdot V + V\cdot\dot u, & (19)\\ \dot e &= -(e+p+\Pi)\,\theta + \pi_{\mu\nu}\nabla^{\langle\mu} u^{\nu\rangle} - \nabla\cdot W + 2\, W\cdot\dot u, & (20)\\ (e+p+\Pi)\,\dot u^\mu &= \nabla^\mu(p+\Pi) - \Delta^{\mu\nu}\nabla^\sigma\pi_{\nu\sigma} + \pi^{\mu\nu}\dot u_\nu - \left[\Delta^{\mu\nu}\dot W_\nu + W^\mu\theta + (W\cdot\nabla)u^\mu\right]. & (21) \end{aligned}$$

The matching conditions (15) leave the choice of the local rest frame velocity $u^\mu$ ambiguous. This ambiguity can be used to eliminate either $V^\mu$ from Eq. (16) ("Eckart frame", no baryon flow in the local rest frame), in which case the energy flow reduces to the heat flow vector $W^\mu = q^\mu$, or $W^\mu$ from Eq. (17) ("Landau frame", no energy flow in the local rest frame), in which case there is a non-zero baryon flow $V^\mu = -\frac{n}{e+p} q^\mu$ due to heat conduction in the local rest frame. (Intermediate frames are also possible, but yield no practical advantage.) For systems with vanishing net baryon number (as approximately realized in RHIC collisions) the Eckart frame is ill-defined, so we will use the Landau frame. In this frame, for baryon-free systems with $n = 0$ and no heat conduction, the non-ideal fluid equations (19-21) simplify to

$$\begin{aligned} \dot e &= -(e+p+\Pi)\,\theta + \pi_{\mu\nu}\nabla^{\langle\mu} u^{\nu\rangle}, & (22)\\ (e+p+\Pi)\,\dot u^\mu &= \nabla^\mu(p+\Pi) - \Delta^{\mu\nu}\nabla^\sigma\pi_{\nu\sigma} + \pi^{\mu\nu}\dot u_\nu. & (23) \end{aligned}$$

The non-equilibrium decompositions (16-18) involve 1+3+5=9 additional dynamical quantities, the "dissipative flows" $\Pi$, $q^\mu$, and $\pi^{\mu\nu}$ (the counting reflects their transversality to $u^\mu$ and the tracelessness of $\pi^{\mu\nu}$). This means that we need 9 additional dynamical equations which should be compatible with the underlying transport theory for the non-equilibrium deviation $\delta f(x,p)$. For the baryon-free case without heat conduction, the number of needed additional equations reduces to 6.

## 4 Kinetic equations for the dissipative flows

The key property of the kinetic equation governing the evolution of the phase-space distribution function $f = f_{\rm eq} + \delta f$ is that the collision term satisfies the second law of thermodynamics (3), i.e. entropy is produced until the system has reached a new state of local thermal equilibrium. We don't want to solve the kinetic theory; instead, we want to write down a phenomenological macroscopic theory which is consistent with the constraints arising from the underlying kinetic theory, in particular the 2$^{\rm nd}$ law. The macroscopic theory will be constructed from an expansion of the entropy production rate in terms of the dissipative flows which themselves are proportional to the off-equilibrium deviation $\delta f$ of the phase-space distribution. Assuming the latter to be small, $|\delta f| \ll |f_{\rm eq}|$, this expansion will be truncated at some low order in the dissipative flows $\delta N^\mu$, $\delta T^{\mu\nu}$. The expansion will involve phenomenological expansion coefficients which, in principle, should be matched to the kinetic theory [3]. In practice, they will often be considered as phenomenological parameters to be adjusted to experimental data. In the end, the extracted values must then be checked for consistency with the entire approach, by making sure that the dissipative corrections are indeed sufficiently small to justify truncation of the expansion *a posteriori*.

The equilibrium identity (11) can be rewritten as

$$S^\mu_{\rm eq} = p(\alpha,\beta)\beta^\mu - \alpha N^\mu_{\rm eq} + \beta_\nu T^{\nu\mu}_{\rm eq}, \tag{24}$$

where $\alpha \equiv \frac{\mu}{T}$, $\beta \equiv \frac{1}{T}$, and $\beta_\nu \equiv \frac{u_\nu}{T}$. The most general off-equilibrium generalization of this is [3]

$$S^\mu \equiv S^\mu_{\rm eq} + \Phi^\mu = p(\alpha,\beta)\beta^\mu - \alpha N^\mu + \beta_\nu T^{\nu\mu} + Q^\mu(\delta N^\mu, \delta T^{\mu\nu}), \tag{25}$$

where, in addition to the first order contributions implicit in the second and third terms of the r.h.s., $Q^\mu$ includes terms which are second and higher order in the dissipative flows $\delta N^\mu$ and $\delta T^{\mu\nu}$. [Note that, by using the identity (11) between the equilibrium quantities, Eq. (25) can be written in the simpler-looking form $S^\mu = s\,u^\mu + \frac{q^\mu}{T} + Q^\mu$ but this is not helpful for calculating the entropy production rate.]

The form of the expansion (25) is constrained by the 2$^{\rm nd}$ law $\partial_\mu S^\mu \geq 0$. To evaluate this constraint it is useful to rewrite the Gibbs-Duhem relation $dp = s\,dT + n\,d\mu$ as

$$\partial_\mu \left(p(\alpha,\beta)\beta^\mu\right) = N^\mu_{\rm eq}\partial_\mu\alpha - T^{\mu\nu}_{\rm eq}\partial_\mu\beta_\nu. \tag{26}$$

With additional help from the conservation laws (1,2), the entropy production then becomes

$$\partial_\mu S^\mu = -\delta N^\mu \partial_\mu \alpha + \delta T^{\mu\nu}\partial_\mu\beta_\nu + \partial_\mu Q^\mu. \tag{27}$$

Using Eqs. (16,17) to express $\delta N^\mu$ and $\delta T^{\mu\nu}$ in terms of the scalar, vector and tensor dissipative flows $\Pi$, $q^\mu$, and $\pi^{\mu\nu}$, and introducing corresponding scalar, vector and tensor thermodynamic forces (in terms of gradients of the thermodynamic equilibrium variables) which drive these dissipative flows, $X \equiv -\theta = -\nabla\cdot u$, $X^\nu \equiv \frac{\nabla^\nu T}{T} - \dot{u}^\nu = -\frac{nT}{e+p}\nabla^\nu(\frac{\mu}{T})$, and $X^{\mu\nu} \equiv \nabla^{\langle\mu}u^{\nu\rangle}$ (note that $X^{\mu\nu} = X^{\langle\mu\nu\rangle}$ is traceless and transverse to $u$), the 2$^{\rm nd}$ law constraint can be further recast into

$$T\partial_\mu S^\mu = \Pi X - q^\mu X_\mu + \pi^{\mu\nu}X_{\mu\nu} + T\partial_\mu Q^\mu \geq 0. \tag{28}$$

Note that the first three terms on the r.h.s. are first order while the last term is higher order in the dissipative flows.

### 4.1 Standard dissipative fluid dynamics (first order theory)

The standard approach (which can be found, for example, in the *Fluid Dynamics* volume of Landau & Lifshitz) one neglects the higher order contributions and sets $Q^\mu = 0$. The inequality (28) can than be always satisfied by postulating linear relationships between the dissipative flows and the thermodynamic forces,

$$\Pi = -\zeta\theta, \qquad q^\nu = -\lambda\frac{nT^2}{e+p}\nabla^\nu\left(\frac{\mu}{T}\right), \qquad \pi^{\mu\nu} = 2\,\eta\,\nabla^{\langle\mu}u^{\nu\rangle}, \tag{29}$$

with positive *transport coefficients* $\zeta \geq 0$ (*bulk viscosity*), $\lambda \geq 0$ (*heat conductivity*), and $\eta \geq 0$ (*shear viscosity*):

$$T\partial\cdot S = \frac{\Pi^2}{\zeta} - \frac{q^\alpha q_\alpha}{2\lambda T} + \frac{\pi^{\alpha\beta}\pi_{\alpha\beta}}{2\eta} \geq 0. \tag{30}$$

(The minus sign in front of the second term is necessary because $q^\mu$, being orthogonal to $u^\mu$, is spacelike, $q^2 < 0$.) These are the desired 9 equations for the dissipative flows.

Unfortunately, using these relations in the hydrodynamic equations (19-21) leads to hydrodynamic evolution with acausal signal propagation: if in a given fluid cell at a certain time a thermodynamic force happens to vanish, the corresponding dissipative flow also stops instantaneously. This contradicts the fact that the flows result from the forces through microscopic scattering which involves relaxation on a finite albeit short kinetic time scale. To avoid this type of acausal behaviour one must keep $Q^\mu$.

### 4.2 Second order Israel-Stewart theory

A causal theory of dissipative relativistic fluid dynamics is obtained by keeping $Q^\mu$ up to terms which are second order in the irreversible flows. For simplicity I here consider only the baryon-free case $n = q^\mu = 0$, for a general treatment see [3, 5]. One writes [3]

$$Q^\mu = -\left(\beta_0\Pi^2 + \beta_2\pi_{\nu\lambda}\pi^{\nu\lambda}\right)\frac{u^\mu}{2T} \tag{31}$$

(with phenomenological expansion coefficients $\beta_0$, $\beta_2$) and computes (after some algebra using similar techniques as before) the entropy production rate as

$$T\partial\cdot S = \Pi\left[-\theta - \beta_0\dot{\Pi} - \Pi T\partial_\mu\left(\frac{\beta_0 u^\mu}{2T}\right)\right] + \pi^{\alpha\beta}\left[\nabla_{\langle\alpha}u_{\beta\rangle} - \beta_2\dot{\pi}_{\alpha\beta} - \pi_{\alpha\beta}T\partial_\mu\left(\frac{\beta_2 u^\mu}{2T}\right)\right]. \tag{32}$$

From the expressions in the square brackets we see that the thermodynamic forces $-\theta$ and $\nabla_{\langle\alpha}u_{\beta\rangle}$ are now self-consistently modified by terms involving the time derivatives (in the local rest frame) of the irreversible flows $\Pi$, $\pi_{\alpha\beta}$. This leads to dynamical ("transport") equations for the latter. We can ensure the $2^{\rm nd}$ law of thermodynamics by again writing the entropy production rate in the form (30) (without the middle term), which amounts to postulating

$$\dot{\Pi} = -\frac{1}{\tau_\Pi}\left[\Pi + \zeta\theta + \Pi\zeta T\partial_\mu\left(\frac{\tau_\Pi u^\mu}{2\zeta T}\right)\right] \approx -\frac{1}{\tau_\Pi}[\Pi + \zeta\theta], \tag{33}$$

$$\dot{\pi}_{\alpha\beta} = -\frac{1}{\tau_\pi}\left[\pi_{\alpha\beta} - 2\eta\nabla_{\langle\alpha}u_{\beta\rangle} + \pi_{\alpha\beta}\eta T\partial_\mu\left(\frac{\tau_\pi u^\mu}{2\eta T}\right)\right] \approx -\frac{1}{\tau_\pi}\left[\pi_{\alpha\beta} - 2\eta\nabla_{\langle\alpha}u_{\beta\rangle}\right]. \tag{34}$$

Here I replaced the coefficients $\beta_{0,2}$ by the relaxation times $\tau_\Pi \equiv \zeta\beta_0$ and $\tau_\pi \equiv 2\eta\beta_2$. In principle both $\zeta, \eta$ and $\tau_\Pi, \tau_\pi$ should be calculated from the underlying kinetic theory. We will use them as phenomenological parameters, noting that for consistency the microscopic relaxation rates should be much larger than the local hydrodynamic expansion rate, $\tau_{\pi,\Pi}\theta \ll 1$.

Let me shortly comment about the approximation in the second equalities in Eqs. (33,34): We are using an expansion scheme for the entropy production rate in which the thermodynamic forces and irreversible flows are assumed to be small perturbations. The approximation in Eqs. (33,34) neglects terms which are products of the irreversible flows with gradients of the thermodynamic equilibrium quantities which are of the same order as the thermodynamic forces. These terms are thus effectively of second order in small quantities and should, for consistency, be neglected relative to the other terms in the square brackets which are of first order. If one wants to keep them (as done by Muronga [5, 6]), one should also keep third-order terms in the entropy flow vector $Q^\mu$ for consistency. Of course, where the thermodynamic forces and irreversible flows are really small, it shouldn't matter whether we keep or drop these terms. In practice, however, one will use this approach when dissipative effects are expected

to be significant, and the dropped terms may not be extremely small. In this case I believe that dropping them is more consistent than keeping them.

There is another reason for dropping these terms: without them, Eqs. (33,34) are relaxation equations which describe (in the local rest frame) exponential relaxation (on the time scales $\tau_{\pi,\Pi}$) of the irreversible flows to the values given by Eqs. (29) in the first order theory. However, if these terms are kept, one has instead equations of the form

$$\dot{\Pi} = -\frac{1}{\tau_\Pi}\Big[\Pi + \zeta\theta + \Pi\zeta\gamma_\Pi\Big] = -\frac{1+\gamma_\Pi\zeta}{\tau_\Pi}\left[\Pi + \frac{\zeta}{1+\gamma_\Pi\zeta}\,\theta\right] = -\frac{1}{\tau'_\Pi}\Big[\Pi + \zeta'\,\theta\Big], \tag{35}$$

and similarly for the shear pressure tensor. One sees that both the kinetic relaxation time and the viscosity are modified by the factor $\gamma_\Pi = T\partial_\mu\left(\frac{\tau_\Pi u^\mu}{2\zeta T}\right)$ which involves the macroscopic expansion rate $\partial_\mu u^\mu$. This violates the intuition that these transport coefficients should be expressible through integrals of the kinetic collision term which involve only microscopic physics (cross sections, local densities, etc.)

In the second order Israel-Stewart formalism, one thus solves the dissipative hydrodynamic equations (19-21) simultaneously with the kinetic relaxation equations (33,34) for the irreversible flows. Let us now look at these equations in more detail when expressed in a global coordinate system (and not in local rest frame coordinates as done up to now).

## 5 Transverse expansion dynamics in central collisions at high energy

I will restrict my discussion here to azimuthally symmetric systems with longitudinal boost invariance. With this approximation we can describe the transverse expansion in central collisions at very high energy in a domain near midrapidity. A generalized discussion which relaxes the assumption of azimuthal symmetry and thus allows for collisions at any impact parameter has been recently given in [9].

Boost-invariant and azimuthally symmetric systems are conveniently described in $(\tau, r, \phi, \eta)$ coordinates where $\tau = \sqrt{t^2 - z^2}$ is longitudinal proper time, $\eta = \frac{1}{2}\ln[(t+z)/(t-z)]$ is space-time rapidity, and $\boldsymbol{r} = (r, \phi)$ are polar coordinates in the plane transverse to the beam direction $z$. Such systems are then characterized by macroscopic observables which are independent of $\phi$ and $\eta$, and by azimuthally constant phase-space distributions which depend only on the difference $Y - \eta$ (where $Y = \frac{1}{2}\ln[(E+p_z)/(E-p_z)]$ is the momentum-space rapidity of a particle with longitudinal momentum $p_z$ and energy $E$). We use lowercase latin letters to denote vector and tensor components in this curvilinear space-time coordinate system. The metric tensor in this coordinate system reads

$$g^{mn} = \mathrm{diag}\,(1, -1, -1/r^2, -1/\tau^2), \qquad g_{mn} = \mathrm{diag}\,(1, -1, -r^2, -\tau^2). \tag{36}$$

The flow velocity is parametrized as

$$u^m = \gamma_r(1, v_r, 0, 0) \quad \text{with} \quad \gamma_r = \frac{1}{\sqrt{1 - v_r^2}} \tag{37}$$

with radial transverse flow velocity $\boldsymbol{v}_\perp = v_r(\tau, r)\,\boldsymbol{e}_r$ and vanishing flow components $u^\phi$ and $u^\eta$. For vectors and tensors, the usual Cartesian derivatives $\partial_\mu$ must be replaced by covariant derivatives, denoted by semicolons:

$$\partial_\mu j^\nu \to j^n_{\;;m} = \partial_m j^n + \Gamma^n_{mk} j^k, \qquad \partial_\mu T^{\nu\lambda} \to T^{nl}_{\;;m} = \partial_m T^{nl} + \Gamma^n_{mk} T^{kl} + T^{nk}\Gamma^l_{km}, \tag{38}$$

where $\Gamma^i_{jk} = \frac{1}{2}g^{im}(\partial_j g_{km} + \partial_k g_{mj} - \partial_m g_{jk})$ are the Christoffel symbols. The nonvanishing components of $\Gamma^i_{jk}$ are

$$\Gamma^\eta_{\eta\tau} = \Gamma^\eta_{\tau\eta} = \frac{1}{\tau}, \quad \Gamma^\tau_{\eta\eta} = \tau, \quad \Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = +\frac{1}{r}, \quad \Gamma^r_{\phi\phi} = -r. \tag{39}$$

The time derivative in the local comoving frame and the local expansion rate are thus computed as

$$D = u \cdot \partial = \gamma_r(\partial_\tau + v_r\partial_r), \tag{40}$$

$$\theta = \partial \cdot u = \frac{1}{\tau}\,\partial_\tau\left(\tau\gamma_r\right) + \frac{1}{r}\,\partial_r\left(rv_r\gamma_r\right) \tag{41}$$

Due to azimuthal symmetry and longitudinal boost invariance, the $n=\phi$ and $n=\eta$ components of the equations of motion $T^{mn}{}_{;m}=0$ are redundant. The $n=\tau$ and $n=r$ components can be written as [6, 9]

$$\frac{1}{\tau}\partial_\tau\Big(\tau T^{\tau\tau}\Big) + \frac{1}{r}\partial_r\Big(rT^{\tau r}\Big) = -\,\frac{p+\Pi+\tau^2\pi^{\eta\eta}}{\tau}, \tag{42}$$

$$\frac{1}{\tau}\partial_\tau\Big(\tau T^{\tau r}\Big) + \frac{1}{r}\partial_r\Big(r(T^{\tau\tau}v_r+\mathcal{P}_r)\Big) = +\,\frac{p+\Pi+r^2\pi^{\phi\phi}}{r}. \tag{43}$$

With the shorthand notations $\tilde{T}^{mn}=r\tau T^{mn}$, $\tilde{\mathcal{P}}_r=r\tau\mathcal{P}_r$, and $\tilde{v}_r=\frac{\tilde{T}^{\tau r}}{\tilde{T}^{\tau\tau}}=\frac{T^{\tau r}}{T^{\tau\tau}}$ these are brought into "standard (Cartesian) form"

$$\partial_\tau\tilde{T}^{\tau\tau} + \partial_r(\tilde{v}_r\tilde{T}^{\tau\tau}) = -r\Big(p+\Pi+\tau^2\pi^{\eta\eta}\Big), \tag{44}$$

$$\partial_\tau\tilde{T}^{\tau r} + \partial_r\Big(v_r\tilde{T}^{\tau r}+\tilde{\mathcal{P}}_r\Big) = +\tau\Big(p+\Pi+r^2\pi^{\phi\phi}\Big). \tag{45}$$

The corresponding transport equations for the dissipative fluxes read

$$\Big(\partial_\tau + v_r\partial_r\Big)\pi^{\eta\eta} = -\frac{1}{\gamma_r\tau_\pi}\left[\pi^{\eta\eta} - \frac{2\eta}{\tau^2}\left(\frac{\theta}{3}-\frac{\gamma_r}{\tau}\right)\right], \tag{46}$$

$$\Big(\partial_\tau + v_r\partial_r\Big)\pi^{\phi\phi} = -\frac{1}{\gamma_r\tau_\pi}\left[\pi^{\phi\phi} - \frac{2\eta}{r^2}\left(\frac{\theta}{3}-\frac{\gamma_r v_r}{r}\right)\right], \tag{47}$$

$$\Big(\partial_\tau + v_r\partial_r\Big)\Pi = -\frac{1}{\gamma_r\tau_{_\Pi}}\left[\Pi + \zeta\theta\right], \tag{48}$$

with the following explicit expressions for the shear tensor components:

$$\sigma^{\eta\eta} = \frac{1}{\tau^2}\left(\frac{\theta}{3}-\frac{\gamma_r}{\tau}\right), \tag{49}$$

$$\sigma^{\phi\phi} = \frac{1}{r^2}\left(\frac{\theta}{3}-\frac{\gamma_r v_r}{r}\right). \tag{50}$$

The hydrodynamic equations require the equation of state (EOS) $p(e)$ for closure, i.e. after each transport step in time we must extract at each spatial grid point the boost velocity $v_r$ between the global and local rest frames and the local energy density $e$ from the dynamical variables $T^{\tau\tau}$ and $T^{\tau r}$. The energy density is obtained from

$$e = T^{\tau\tau} - v_r T^{\tau r}, \tag{51}$$

where the radial velocity $v_r$ must be extracted from the implicit equation

$$v_r = \frac{T^{\tau r}}{T^{\tau\tau} + p(e{=}T^{\tau\tau}{-}v_rT^{\tau r}) + \Pi - r^2\pi^{\phi\phi} - \tau^2\pi^{\eta\eta}} \tag{52}$$

by a one-dimensional zero search. This is still the same degree of numerical complexity as in the ideal fluid case [8]; for dissipative hydrodynamics *without* azimuthal symmetry, however, this part of the problem becomes numerically more involved [9].

## 6 First numerical results

I close this talk by showing some preliminary results [7] from a numerical simulation of the equations derived in the preceding section, using a simple massless ideal gas EOS, $p = \frac{1}{3}e$, with $e = aT^4$, $a = (16 + \frac{21}{2}N_f)\frac{\pi^2}{30}$. We neglect bulk viscosity, $\zeta = 0$.

In classical kinetic theory, explicit expressions can be obtained for the viscosity coefficient $\eta$ and the relaxation time $\tau_\pi$ in terms of the collision term. For a strongly coupled QGP, neither $\eta$ or $\tau_\pi$ are known. We treat them as phenomenological parameters. For guidance, we use perturbative [10, 11] and AdS/CFT-based [12] estimates for $\eta$, respectively, and a kinetic theory estimate [3] for $\tau_\pi$.

The shear viscosity coefficient $\eta$ for hot QCD was determined perturbatively to leading logarithmic accuracy in [10, 11]. For $\alpha_s \approx 0.5$ the result in [11] gives $\frac{\eta}{s} = 0.135$. A lower limit for the shear viscosity in infinitely strongly coupled $N = 4$ SUSY YM theory and variations thereof was derived in [12], exploiting the AdS/CFT correspondence: $\frac{\eta}{s} \geq \frac{1}{4\pi} = 0.08$. In kinetic theory, in the Boltzmann gas approximation, the relaxation time is estimated as $\tau_\pi = 2\eta\beta_2 = 2\eta\,\frac{3}{4p}$ [3].

For the initial energy density distribution in the transverse plane, we used a Woods-Saxon parameterisation,

$$e(r) = \frac{e_0}{1 + e^{\frac{r-R}{a}}}, \tag{53}$$

with $R = 6.4$ fm, $a = 0.54$ fm. This is not very realistic, but facilitates comparison with the results of [6]. ($e_0 = aT_i^4$ is the central energy density at initial time $\tau = \tau_i$.) I show results for initial conditions $T_i = 0.3$ GeV and $\tau_i = 0.5$ fm/$c$, with zero initial radial flow ($v_r(r, \tau_i) = 0$).

For the non-ideal fluid, initial viscous pressures $\pi^{rr}_{\rm ini}$ and $\pi^{\phi\phi}_{\rm ini}$ are required. Even though $v_r$ and its derivatives are zero initially, due to the Bjorken longitudinal motion the stress tensor is not zero: $\tau_i^2\sigma^{\eta\eta} = -\frac{2}{3\tau_i}$, $r^2\sigma^{\phi\phi} = \frac{1}{3\tau_i}$. We assume that at initial time $\tau_i$, the viscous pressure components are fully relaxed to the Bjorken scaling expansion values,

$$\pi^{rr}_{\rm ini} = r^2\pi^{\phi\phi}_{\rm ini} = -\frac{\tau_i^2}{2}\pi^{\eta\eta}_{\rm ini} = \frac{2\eta}{3\tau_i}. \tag{54}$$

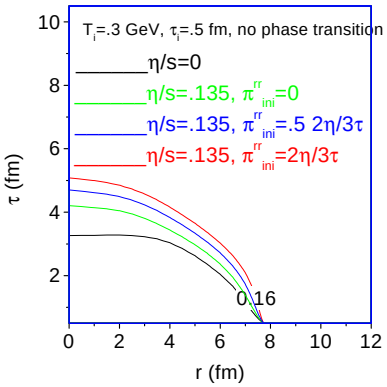


Fig. 1: $T_f$=160 MeV surface for different initial viscous pressures.

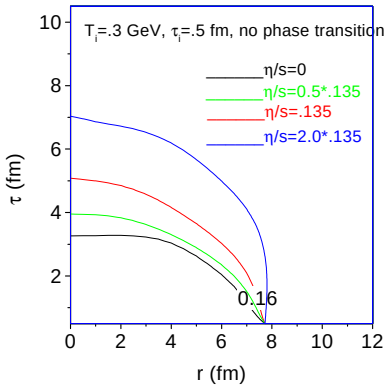


Fig. 2: $T_f = 160$ MeV surface for different shear viscosities $\eta$.

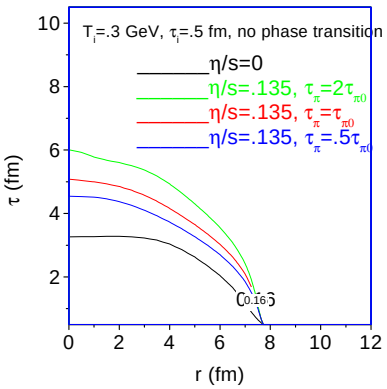


Fig. 3: $T_f$=160 MeV surface for different relaxation times $\tau_\pi$.

Figures 1 to 3 show that the space-time evolution of non-ideal fluids depends sensitively on (i) the initial viscous pressure, (ii) the viscosity coefficient $\eta$, and (iii) the relaxation time $\tau_\pi$. In all three figures we show the shape of the the $T_f = 0.16$ GeV surface, with the solid black line giving the ideal fluid result for reference. Viscous effects are seen to slow down the cooling of the matter, increasing the lifetime of the fireball and its average transverse size at freeze-out.

In Fig. 1 the $T_f = 0.16$ GeV surface is shown for different initial viscous pressures, $\pi^{rr}_{\rm ini} = 0$, $\frac{\eta}{3\tau_i}$ and $\frac{4\eta}{3\tau_i}$, respectively, using $\frac{\eta}{s} = 0.135$. The higher the initial viscous pressure, the more extended is the freeze-out surface and the larger are the deviations from ideal fluid dynamics. The life time of the dissipative QGP is extended by 20% if the initial viscous pressure is increased from zero to $\frac{4\eta}{3\tau_i}$. The freeze-out surface also depends sensitively on the value of the viscosity coefficient (Fig. 2). As the viscosity decreases, the departure of the freeze-out surface from ideal behavior also decreases. In Fig. 3 we show the freeze-out surface for different relaxation times, $\tau_\pi = 0.5\,\tau_{\rm kin}$, $\tau_{\rm kin}$ and $2\,\tau_{\rm kin}$ (where $\tau_{\rm kin} = \frac{3\eta}{2p}$), for fixed viscosity $\eta/s = 0.135$. As the relaxation time is increased by a factor 4, the freeze-out time in the fireball center increases by 25%.

All the viscous effects shown in Figures 1 to 3 increase if the initial time $\tau_i$ is decreased, keeping the total fireball entropy and all other parameters constant. This is due to the increasing initial longitudinal expansion rate $\theta = \frac{1}{\tau_i}$ which results in a larger ratio $\frac{\tau_\pi}{\tau_i}$. This ratio is the figure of merit which controls the importance of viscous corrections to ideal fluid dynamics. Our studies show that there is not only a minimum thermalization time $\tau_{\rm therm}$ after which ideal fluid dynamic can be applied, but there is also a minimum time $\tau_i < \tau_{\rm therm}$ for the applicability of viscous fluid dynamics. The initial conditions for viscous hydrodynamics at that time $\tau_i$ must be obtained by matching the decomposition (17) of the energy-momentum tensor to the corresponding result for $T^{\mu\nu}$ from some preceding non-equilibrium kinetic evolution.

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